Attempting Induction on the Collatz Conjecture

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Abstract

An *incomplete* proof of the Collatz Conjecture by induction is presented along with identification and discussion of gaps and difficulties with the approach. It is presented without expectation of ever being completed, but rather as an instructive alternate way of thinking about the Collatz Conjecture.

1 Introduction

The Collatz Conjecture was first proposed in 1937 by Lothar Collatz. The Conjecture pertains to a particular sequence that can be defined for any natural number.

Definition 1. *Let $c \in \mathbb{N}$. The Collatz sequence for c begins with c. The next term is always found by operating on the previous term. If the previous term is odd, the next term in the sequence is triple the previous term plus one. If the previous term is even, the next term is half the previous term.*

Collatz conjectured that the Collatz sequence for every natural number eventually reaches 1.

2 The Induction Proof Attempt

A number of results can be built up that result in limiting the scope of the induction proof. The basic approach is to limit the proof to only one equivalence class mod 4. To achieve this, we will first show that every Collatz sequence ultimately joins another Collatz sequence that begins with a member of the equivalence class $[1]_4$.

Note that the equivalence classes used in this paper are derived from the standard congruence modulo n relation on the natural numbers.

Lemma 1. *Collatz sequences of the form $c = 2^k$ satisfy the Conjecture for $k \geq 1$.*

Proof. Dividing c repeatedly by 2 results in the Collatz sequence arriving at 1 and so the Collatz sequence for$c$ satisfies the Conjecture. □

Lemma 2. *If the Collatz sequence for c satisfies the Conjecture, then the Collatz sequence for $\hat{c} = 2^k \cdot c$ also satisfies the Conjecture.*

Proof. Dividing $\hat{c}$ repeatedly results in the Collatz sequence arriving at c whose Collatz sequence has been assumed to satisfy the Conjecture. Therefore, the Collatz sequence for $\hat{c}$ satisfies the Conjecture. □

Lemma 3. *If c is odd and its Collatz sequence satisfies the Conjecture, then the Collatz sequence for $4c + 1$ also satisfies the Conjecture.*

Proof. By Lemma 2, the Collatz sequence for $4c = 2^2 \cdot c$ satisfies the Conjecture. The Collatz sequence for $4c$ begins with $c_0 = 4c$, $c_1 = 2c$, $c_2 = c$ (odd), $c_3 = 3c+1$. The Collatz sequence for $4c+1$ begins with $c_0 = 4c+1$ (odd), $c_1 = 12c+4$, $c_2 = 6c + 2$, $c_3 = 3c + 1$ which joins the Collatz sequence for $4c$ at c_3. Since the Collatz sequence for $4c$ satisfies the Conjecture, the Collatz sequence for $4c + 1$ satisfies the Conjecture as well. □

Lemma 4. *If the Collatz sequence for $c = 4k + 1$ satisfies the Conjecture with k even, then the Collatz sequence for $c' = 12k + 5$ also satisfies the Conjecture.*

Proof. Let $c = 4k + 1$ with k even. The Collatz sequence for c begins with $c_0 = 4k + 1$, $c_1 = 12k + 4$, $c_2 = 6k + 2$, and $c_3 = 3k + 1$. Since $3k + 1$ is a member of a Collatz sequence that satisfies the Conjecture, the Collatz sequence for $3k + 1$ also satisfies the Conjecture. Additionally, k is even, so $3k + 1$ is an odd number whose Collatz sequence that satisfies the Conjecture. By Lemma 3, $c' = 12k + 5 = 4(3k + 1) + 1$ must also satisfy the Conjecture. □

Lemma 5. *The union*

$$\bigcup_{k=2}^{n} [2^{k-1} - 1]_{2^k}$$

of equivalence classes forms a covering of the odd members of $\mathbb{N} \setminus [2^n - 1]_{2^n}$.

Proof. Let $n = 2$. Then

$$\bigcup_{k=2}^{2} [2^{k-1} - 1]_{2^k} = [1]_4$$

which fails to cover the odd numbers that are members of the equivalence class $[3]_4$. For $n = 3$,

$$\bigcup_{k=2}^{3} [2^{k-1} - 1]_{2^k} = [1]_4 \cup [3]_8 = [1]_8 \cup [5]_8 \cup [3]_8$$

which fails to cover the odd numbers that are members of the equivalence class $[7]_8$. Continuing the pattern, it is clear that for a given n the equivalence classes from the previous step will cover all but two classes of odd numbers: $[2^n - 1]_{2^n}$ and $[2^{n-1} - 1]_{2^n}$. The latter is covered by the current step leaving the former as the only uncovered equivalence class. The result therefore holds by induction. □

Lemma 6. *The infinite union*

$$\lim_{n\to\infty} \bigcup_{k=2}^{n} [2^{k-1}-1]_{2^k}$$

is a covering of the odd members of $\mathbb{N}$.

Proof. For $n = 1$, $n \in [1]_4$. For $n = 3$, $n \in [3]_8$. Now let $n \in \mathbb{N}$ be an odd number greater than 3 and let $m \in \mathbb{N}$ be the largest number such that $2^m < n$, $m \geq 2$. Then $n = 2^m + p$ for some $p \in \{1, 3, ..., 2^m - 1\}$, i.e. $n \in [p]_{2^m}$. If $p \neq 2^m - 1$, then by Lemma 5

$$n \in \bigcup_{k=2}^{m} [2^{k-1}-1]_{2^k}$$

If $p = 2^m - 1$, then since m was the largest number such that $2^m < n$,

$$n = 2^m + p = 2^m + (2^m - 1) = 2 \cdot 2^m - 1 = 2^{m+2} \cdot 0 + (2^{m+1} - 1) \in [2^{m+1} - 1]_{2^{m+2}}$$

and the result holds. □

Lemma 7. *The Collatz sequence for every odd* $c > 3$ *contains a member of* $[1]_4$ *as one of its terms.*

Proof. Let $c \in \mathbb{N}$ be odd with $c > 3$. The by Lemma 6 $\exists k \in \mathbb{N}$ such that $c \in [2^{k-1} - 1]_{2^k}$. So, $\exists n \in \mathbb{N}$ such that $c = 2^k \cdot n + (2^{k-1} - 1)$. The Collatz sequence for c begins with

$$\begin{aligned}
c_0 &= 2^k \cdot n + (2^{k-1} - 1) = 2^{k-1}(2n+1) - 1 \\
c_1 &= 3 \cdot 2^{k-1}(2n+1) - 2 \\
c_2 &= 3 \cdot 2^{k-2}(2n+1) - 1 \\
c_3 &= 3^2 \cdot 2^{k-2}(2n+1) - 2 \\
c_4 &= 3^2 \cdot 2^{k-3}(2n+1) - 1 \\
&\vdots \\
c_{2k-4} &= 3^{k-2} \cdot 2^1(2n+1) - 1 \\
&= 3^{k-2} \cdot 4n + 2 \cdot 3^{k-2} - 1 \\
&= 4(3^{k-2} \cdot n + \frac{3^{k-2} - 1}{2}) + 1
\end{aligned}$$

noting that $3^{k-2} - 1$ is even for $k > 2$. Therefore, $c_{2k-4} \in [1]_4$ and the result is established.

□

By Lemma 7, every Collatz sequence eventually joins another Collatz sequence that begins with a member of the equivalence class $[1]_4$. Therefore, it is sufficient to prove the Collatz Conjecture only for members of $[1]_4$.

Now we can attempt the induction proof of the Collatz Conjecture on this one equivalence class. In the cases covered by the proof it is made very clear where the gaps and shortcomings are. The discussion afterwards makes it clear that this method of proof has virtually no chance of ultimately succeeding. But, the proof itself reveals the great amount of structure and patterns which exist in Collatz sequences and how "few" of the sequences actually prevent the proof from succeeding, i.e. how close the induction proof comes to succeeding.

Theorem 1. *(Collatz) The Collatz sequence for every member of the equivalence class* $[1]_4$ *eventually reaches 1.*

Proof. Consider the partition of the natural numbers modulo 64. Then the members of the equivalence class $[1]_4$ are members of $[1]_{64}$, $[5]_{64}$, $[9]_{64}$, ..., $[61]_{64}$.

The Collatz sequences for 1 and 5 satisfy the Conjecture:

$$c = 1 : 1, 4, 2, 1$$
$$c = 5 : 5, 16, 8, 4, 2, 1$$

Let $c > 5$ be an arbitrary member of $[1]_4$, and assume the Collatz sequence for each member of $[1]_4$ less than c satisfies the Conjecture.

Case 1: $c \in [1]_{16}$. Then $\exists n \in \mathbb{N}$ such that $c = 16n+1 = 4(4n)+1$. Note that we have assumed that the Collatz sequence for $4k+1$ satisfies the Conjecture for every $k < 4n$. Now, the Collatz sequence for $16n + 1$ begins with $c_0 = 16n + 1$, $c_1 = 48n + 4$, $c_2 = 24n + 2$, $c_3 = 12n + 1 = 4(3n) + 1$. Since $3n < 4n$, the Collatz sequence for $4(3n) + 1$ has been assumed to satisfy the Conjecture. Therefore, the Collatz sequence for members of $[1]_{16}$ joins at term c_3 a sequence that satisfies the Conjecture. Note that this covers the equivalence classes $[1]_{64}$, $[17]_{64}$, $[33]_{64}$, and $[49]_{64}$.

Case 2: $c \in [5]_{16}$. Then $\exists n \in \mathbb{N}$ such that $c = 16n+5 = 4(4n+1)+1$. Since $\hat{c} = 4n + 1 < c$, it's Collatz sequence satisfies the Conjecture by assumption. Since $4n + 1$ is odd, by Lemma 3, the Collatz sequence for $4(4n + 1) + 1 = 16n + 5 = c$ also satisfies the Conjecture. Note that this covers the equivalence classes $[5]_{64}$, $[21]_{64}$, $[37]_{64}$, and $[53]_{64}$.

Case 3: $c \in [9]_{64}$. Sub-cases $[9]_{2048}$, $[73]_{128}$, $[265]_{1024}$, $[393]_{1024}$, $[521]_{1024}$, $[649]_{2048}$, and $[1033]_{2048}$ satisfy the Conjecture based on Results 3a through 3g in the "Supporting Results" section, which leaves only sub-cases $[137]_{1024}$, $[777]_{1024}$ (solved at modulus 4096), $[905]_{1024}$, and $[1673]_{2048}$ as **unresolved**.

Case 4: $c \in [13]_{32}$. Then $\exists n \in \mathbb{N}$ such that $c = 32n + 13 = 4(8n + 3) + 1$. Note that we have assumed that the Collatz sequence for $4k + 1$ satisfies the Conjecture for every $k < 8n + 3$. Now, the Collatz sequence for $32n + 13$ begins with $c_0 = 32n + 13$, $c_1 = 96n + 40$, $c_2 = 48n + 20$, $c_3 = 24n + 10$, $c_4 = 12n + 5 = 4(3n + 1) + 1$. Since $3n + 1 < 8n + 3$, the Collatz sequence for $4(3n + 1) + 1$ has been assumed to satisfy the Conjecture. Therefore, the

Collatz sequence for members of $[13]_{32}$ joins at term c_4 a sequence that satisfies the Conjecture. Note that this covers the equivalence classes $[13]_{64}$ and $[45]_{64}$.

Case 5: $c \in [25]_{32}$. Then $\exists n \in \mathbb{N}$ such that $c = 32n+25 = 4(8n+6)+1$. Note that we have assumed that the Collatz sequence for $4k+1$ satisfies the Conjecture for every $k < 8n+6$. Consider the Collatz sequence for $\hat{c} = 12n+9 = 4(3n+2)+1$ which has been assumed to satisfy the Conjecture. The sequence begins with $c_0 = 12n+9$, $c_1 = 36n+28$, $c_2 = 18n+14$, $c_3 = 9n+7$. Since $9n+7$ is odd and its Collatz sequence satisfies the Conjecture, by Lemma 3 the Collatz sequence for $4(9n+7)+1 = 36n+29$ also satisfies the Conjecture.

Now, the Collatz sequence for $32n+25$ begins with $c_0 = 32n+25$, $c_1 = 96n+76$, $c_2 = 48n+38$, $c_3 = 24n+19$, $c_4 = 72n+58$, $c_5 = 36n+29$. As shown above, the Collatz sequence for $36n+29$ satisfies the Conjecture. Therefore, the Collatz sequence for members of $[25]_{32}$ joins at term c_5 a sequence that satisfies the Conjecture. Note that this covers the equivalence classes $[25]_{64}$ and $[57]_{64}$.

Case 6: $c \in [29]_{64}$. Then $\exists n \in \mathbb{N}$ such that $c = 64n+29 = 4(16n+7)+1$. Note that we have assumed that the Collatz sequence for $4k+1$ satisfies the Conjecture for every $k < 16n+7$. Now, the Collatz sequence for $62n+29$ begins with $c_0 = 64n+29$, $c_1 = 192n+88$, $c_2 = 96n+44$, $c_3 = 48n+22$, $c_4 = 24n+11$, $c_5 = 72n+34$, $c_6 = 36n+17 = 4(9n+4)+1$. Since $9n+4 < 16n+7$, the Collatz sequence for $4(9n+4)+1$ has been assumed to satisfy the Conjecture. Therefore, the Collatz sequence for members of $[29]_{64}$ joins at term c_6 a sequence that satisfies the Conjecture.

Case 7: $c \in [41]_{64}$. Sub-cases $[105]_{256}$, $[489]_{2048}$, and $[809]_{1024}$ satisfy the Conjecture based on Results 7a through 7c in the "Supporting Results" section, which leaves sub-cases $[41]_{512}$, $[169]_{256}$, $[297]_{1024}$, $[233]_{512}$, $[1001]_{1024}$, and $[1513]_{2048}$ as **unresolved**.

Case 8: $c \in [61]_{64}$. Sub-cases $[61]_{128}$, $[125]_{1024}$, $[253]_{512}$, and $[381]_{512}$ satisfy the Conjecture based on Results 8a through 8d in the "Supporting Results" section, which leaves only sub-cases $[509]_{512}$ and $[637]_{1024}$ as **unresolved**. □

3 Discussion

The induction "proof" presented above centers on members of the equivalence class $[1]_4$. If the modulus of this equivalence class is changed to 2048 (the highest modulus used in the proof to prove convergence of a subset of $[1]_4$), then the equivalence class $[1]_4$ is replaced by 512 equivalence classes. The proof manages to show that 478 of these equivalence classes satisfy the Conjecture under an induction proof approach. This means that only about 6-7% of the equivalence classes remain to be "solved".

To illustrate the difficulty in making further progress on the induction proof, consider the equivalence class $[509]_{512}$ which was left unproven in Case 8. The Collatz sequence for 509 suffers 14 divisions by 2 before it becomes less than the starting value and attains a number that is a member of $[1]_4$ (the number 205 that is the 23rd member of the sequence). This would mean that members of $[509]_{65536}$ can be shown to satisfy the Conjecture. But, this has increased the

modulus by a factor of $2^5 = 32$ from 2048 to 65536 which means that a whole host of other equivalence classes must be investigated to "fill in the holes" (every time the modulus is doubled the number of equivalence classes doubles).

While every increase in modulus to solve a particular subset causes a decrease in the percentage of equivalence classes that remain unsolved, the actual number of unsolved equivalence classes continues to increase. Effectively, in order for the induction proof to complete, a ceiling would have to exist for the number of divisions by 2 necessary for Collatz sequences of members of $[1]_4$ to reach a smaller member of $[1]_4$. *There is no evidence that such a ceiling exists* which is why a proof of the Collatz Conjecture by this type of induction is almost certainly impossible.

What this induction proof attempt reinforces is that there are many, many patterns and a great deal of structure among Collatz sequences, to the point that one could even say a "vast majority" of the sequences are "tame." But sadly, there always appear to be curious, stubbornly resistant Collatz sequences at every level of analysis that preclude a complete justification of the Conjecture.

4 Supporting Results

Result 3a. *The Collatz sequence for members c of the equivalence class $[9]_{2048}$ joins the Collatz sequence for a number $c' < c$ where $c' \in [1]_4$.*

Proof. Let $c \in [9]_{2048}$. Then $\exists n \in \mathbb{N}$ such that $c = 2048n + 9$. The Collatz sequence for c proceeds as:

$$
\begin{aligned}
c_0 &= 2048n + 9 = 4(512n + 2) + 1 \\
c_1 &= 6144n + 28 \\
c_2 &= 3072n + 14 \\
c_3 &= 1536n + 7 \\
c_4 &= 4608n + 22 \\
c_5 &= 2304n + 11 \\
c_6 &= 6912n + 34 \\
c_7 &= 3456n + 17 \\
c_8 &= 10368n + 52 \\
c_9 &= 5184n + 26 \\
c_{10} &= 2592n + 13 \\
c_{11} &= 7776n + 40 \\
c_{12} &= 3888n + 20 \\
c_{13} &= 1944n + 10 \\
c_{14} &= 972n + 5 = 4(243n + 1) + 1
\end{aligned}
$$

Since $243n + 1 < 512n + 2$, the Collatz sequence for c joins a Collatz sequence for a member of $[1]_4$ that is smaller than c. □

Result 3b. *Let $c \in [1]_4$ also be a member of the equivalence class $[73]_{128}$ and assume the Collatz sequence for smaller members of $[1]_4$ satisfy the Conjecture. Then the Collatz sequence for members of $[73]_{128}$ satisfy the Conjecture.*

Proof. Let $c \in [73]_{128}$. Then $\exists k \in \mathbb{N}$ such that $c = 128k + 73$. The Collatz sequence for c proceeds as:

$$\begin{aligned}
c_0 &= 128k + 73 = 4(32k + 18) + 1 \\
c_1 &= 384k + 220 \\
c_2 &= 192k + 110 \\
c_3 &= 96k + 55 \\
c_4 &= 288k + 166 \\
c_5 &= 144k + 83 \\
c_6 &= 432k + 250 \\
c_7 &= 216k + 125 = 12(18k + 10) + 5
\end{aligned}$$

Since $18k + 10 < 32k + 18$, the Collatz sequence for $4(18k + 10) + 1$ is assumed to satisfy the Conjecture. Therefore, the member c_7 of the sequence is of the form required by Lemma 4. Therefore, the Collatz sequence for c joins a Collatz sequence that satisfies the Conjecture and so itself satisfies the Conjecture as well. □

Result 3c. *The Collatz sequence for members c of the equivalence class $[265]_{1024}$ joins the Collatz sequence for a number $c' < c$ where $c' \in [1]_4$.*

Proof. Let $c \in [265]_{1024}$. Then $\exists n \in \mathbb{N}$ such that $c = 1024n + 265$. The Collatz sequence for c proceeds as:

$$\begin{aligned}
c_0 &= 1024n + 265 = 4(256n + 66) + 1 \\
c_1 &= 3072n + 796 \\
c_2 &= 1536n + 398 \\
c_3 &= 768n + 199 \\
c_4 &= 2304n + 598 \\
c_5 &= 1152n + 299 \\
c_6 &= 3456n + 898 \\
c_7 &= 1728n + 449 \\
c_8 &= 5184n + 1348 \\
c_9 &= 2592n + 674 \\
c_{10} &= 1296n + 337 \\
c_{11} &= 3888n + 1012 \\
c_{12} &= 1944n + 506 \\
c_{13} &= 972n + 253 = 4(243n + 63) + 1
\end{aligned}$$

Since $243n+63 < 256n+66$, the Collatz sequence for c joins a Collatz sequence for a member of $[1]_4$ that is smaller than c. □

Result 3d. *Let $c \in [1]_4$ also be a member of the equivalence class $[393]_{1024}$ and assume the Collatz sequence for smaller members of $[1]_4$ satisfy the Conjecture. Then the Collatz sequence for members of $[393]_{1024}$ satisfy the Conjecture.*

Proof. Let $c \in [393]_{1024}$. Then $\exists k \in \mathbb{N}$ such that $c = 1024k + 393$. The Collatz sequence for c proceeds as:

$$\begin{aligned}
c_0 &= 1024k + 393 = 4(256k + 98) + 1 \\
c_1 &= 3072k + 1180 \\
c_2 &= 1536k + 590 \\
c_3 &= 768k + 295 \\
c_4 &= 2304k + 886 \\
c_5 &= 1152k + 443 \\
c_6 &= 3456k + 1330 \\
c_7 &= 1728k + 665 \\
c_8 &= 5184k + 1996 \\
c_9 &= 2592k + 998 \\
c_{10} &= 1296k + 499 \\
c_{11} &= 3888k + 1498 \\
c_{12} &= 1944k + 749 = 12(162k + 62) + 5
\end{aligned}$$

Since $162k+62 < 256k+98$, the Collatz sequence for $4(162k+62)+1$ is assumed to satisfy the Conjecture. Therefore, the member c_{12} of the sequence is of the form required by Lemma 4. Therefore, the Collatz sequence for c joins a Collatz sequence that satisfies the Conjecture and so itself satisfies the Conjecture as well. □

Result 3e. *Let $c \in [1]_4$ also be a member of the equivalence class $[521]_{1024}$ and assume the Collatz sequence for smaller members of $[1]_4$ satisfy the Conjecture. Then the Collatz sequence for members of $[521]_{1024}$ satisfy the Conjecture.*

Proof. Let $c \in [521]_{1024}$. Then $\exists n \in \mathbb{N}$ such that $c = 1024n + 521$. The Collatz

sequence for c proceeds as:

$$\begin{aligned}
c_0 &= 1024n + 521 = 4(256n + 130) + 1 \\
c_1 &= 3072n + 1564 \\
c_2 &= 1536n + 782 \\
c_3 &= 768n + 391 \\
c_4 &= 2304n + 1174 \\
c_5 &= 1152n + 587 \\
c_6 &= 3456n + 1762 \\
c_7 &= 1728n + 881 \\
c_8 &= 5184n + 2644 \\
c_9 &= 2592n + 1322 \\
c_{10} &= 1296n + 661 = 4[4(81n + 41) + 1] + 1
\end{aligned}$$

Since $81n + 41 < 256n + 130$, the Collatz sequence for $4(81n + 41) + 1$ satisfies the Conjecture. Additionally, $4(81n + 41) + 1$ is odd so by Lemma 3 the Collatz sequence for $c_{10} = 4[4(81n + 41) + 1] + 1$ satisfies the Conjecture. Therefore, the Collatz sequence for c satisfies the Conjecture. □

Result 7a. *Let $c \in [1]_4$ also be a member of the equivalence class $[105]_{256}$ and assume the Collatz sequence for smaller members of $[1]_4$ satisfy the Conjecture. Then the Collatz sequence for members of $[105]_{256}$ satisfy the Conjecture.*

Proof. Let $c \in [105]_{256}$. Then $\exists k \in \mathbb{N}$ such that $c = 256k + 105$. The Collatz sequence for c proceeds as:

$$\begin{aligned}
c_0 &= 256k + 61 = 4(64k + 26) + 1 \\
c_1 &= 768k + 184 \\
c_2 &= 384k + 92 \\
c_3 &= 192k + 46 \\
c_4 &= 576k + 23 \\
c_5 &= 288k + 70 \\
c_6 &= 864k + 35 \\
c_7 &= 432k + 106 \\
c_8 &= 1296k + 53 \\
c_9 &= 648k + 269 = 12(54k + 22) + 5
\end{aligned}$$

Since $54k + 22 < 64k + 26$, the Collatz sequence for $4(54k + 22) + 1$ is assumed to satisfy the Conjecture. Therefore, the member c_9 of the sequence is of the form required by Lemma 4. Therefore, the Collatz sequence for c joins a Collatz sequence that satisfies the Conjecture and so itself satisfies the Conjecture as well. □

Result 8a. *The Collatz sequence for members c of the equivalence class $[61]_{128}$ joins the Collatz sequence for a number $c' < c$ where $c' \in [1]_4$.*

Proof. Let $c \in [61]_{128}$. Then $\exists k \in \mathbb{N}$ such that $c = 128k + 61$. The Collatz sequence for c proceeds as:

$$\begin{aligned}
c_0 &= 128k + 61 = 4(32k + 15) + 1 \\
c_1 &= 384k + 184 \\
c_2 &= 192k + 92 \\
c_3 &= 96k + 46 \\
c_4 &= 48k + 23 \\
c_5 &= 144k + 70 \\
c_6 &= 72k + 35 \\
c_7 &= 216k + 106 \\
c_8 &= 108k + 53 = 4(27k + 13) + 1
\end{aligned}$$

Therefore, the Collatz sequence for c joins a Collatz sequence for a member of $[1]_4$ that is smaller than c. □

Result 8b. *The Collatz sequence for members c of the equivalence class $[125]_{1024}$ joins the Collatz sequence for a number $c' < c$ where $c' \in [1]_4$.*

Proof. Let $c \in [125]_{1024}$. Then $\exists k \in \mathbb{N}$ such that $c = 1024k + 125$. The Collatz sequence for c proceeds as:

$$\begin{aligned}
c_0 &= 1024k + 125 = 4(256k + 31) + 1 \\
c_1 &= 3072k + 376 \\
c_2 &= 1536k + 188 \\
c_3 &= 768k + 94 \\
c_4 &= 384k + 47 \\
c_5 &= 1152k + 142 \\
c_6 &= 576k + 71 \\
c_7 &= 1728k + 214 \\
c_8 &= 864k + 107 \\
c_9 &= 2592k + 322 \\
c_{10} &= 1296k + 161 \\
c_{11} &= 3888k + 484 \\
c_{12} &= 1944k + 242 \\
c_{13} &= 972k + 121 = 4(243k + 30) + 1
\end{aligned}$$

Therefore, the Collatz sequence for c joins a Collatz sequence for a member of $[1]_4$ that is smaller than c. □

Result 8c. *The Collatz sequence for members c of the equivalence class $[253]_{512}$ joins the Collatz sequence for a number $c' < c$ where $c' \in [1]_4$.*

Proof. Let $c \in [253]_{512}$. Then $\exists k \in \mathbb{N}$ such that $c = 512k + 253$. Consider the Collatz sequence for c and the Collatz sequence for members of $[125]_{256}$:

$$
\begin{aligned}
c_0 &= 512k + 253 = 4(128k + 63) + 1 \\
c_1 &= 1536k + 760 \\
c_2 &= 768k + 380 \\
c_3 &= 384k + 190 \\
c_4 &= 192k + 95 \\
c_5 &= 576k + 286 \\
c_6 &= 288k + 143 \\
c_7 &= 864k + 430 \\
c_8 &= 432k + 215 \\
c_9 &= 1296k + 646 \\
c_{10} &= 648k + 323 \\
c_{11} &= 1944k + 970 \\
c_{12} &= 972k + 485 \\
c_{13} &= 2916 + 1456 \\
c_{14} &= 1458k + 728 \\
c_{15} &= 729k + 364
\end{aligned}
\qquad
\begin{aligned}
c_0 &= 256k + 125 = 4(64k + 31) + 1 \\
c_1 &= 768k + 376 \\
c_2 &= 384k + 188 \\
c_3 &= 192k + 94 \\
c_4 &= 96k + 47 \\
c_5 &= 288k + 142 \\
c_6 &= 144k + 71 \\
c_7 &= 432k + 214 \\
c_8 &= 216k + 107 \\
c_9 &= 648k + 322 \\
c_{10} &= 324k + 161 \\
c_{11} &= 972k + 484 \\
c_{12} &= 486k + 242 \\
c_{13} &= 243k + 121 \\
c_{14} &= 729k + 364
\end{aligned}
$$

Therefore the Collatz sequence for members of $c \in [253]_{512}$ joins a Collatz sequence for a member of $[1]_4$ that is smaller than c. □

Result 8d. *The Collatz sequence for members c of the equivalence class $[381]_{512}$ joins the Collatz sequence for a number $c' < c$ where $c' \in [1]_4$.*

Proof. Let $c \in [381]_{512}$. Then $\exists k \in \mathbb{N}$ such that $c = 512k + 381$. Consider the Collatz sequence for c and the Collatz sequence for members of $[189]_{256}$:

$c_0 = 512k + 381 = 4(128k + 95) + 1$
$c_1 = 1536k + 1144$
$c_2 = 768k + 572$
$c_3 = 384k + 286$
$c_4 = 192k + 143$
$c_5 = 576k + 430$
$c_6 = 288k + 215$
$c_7 = 864k + 646$
$c_8 = 432k + 323$
$c_9 = 1296k + 970$
$c_{10} = 648k + 485$
$c_{11} = 1944k + 1456$
$c_{12} = 972k + 728$
$c_{13} = 486k + 364$

$c_0 = 256k + 189 = 4(64k + 47) + 1$
$c_1 = 768k + 568$
$c_2 = 384k + 284$
$c_3 = 192k + 142$
$c_4 = 96k + 71$
$c_5 = 288k + 214$
$c_6 = 144k + 107$
$c_7 = 432k + 322$
$c_8 = 216k + 161$
$c_9 = 648k + 484$
$c_{10} = 324k + 242$
$c_{11} = 162k + 121$
$c_{12} = 486k + 364$

Therefore the Collatz sequence for members of $c \in [381]_{512}$ joins a Collatz sequence for a member of $[1]_4$ that is smaller than c. □

About the author

Kevin received a Bachelor of Science degree in Mathematics (minor in Physics) from Sam Houston State University in 1995 with a Master of Science degree in Mathematics following from Oregon State University in 1997. Most recently Kevin passed the first 5 actuarial exams. His Master's thesis focused on taxicab geometry, a field in which he has written 6 research papers, three of which have been published in peer-reviewed journals. He began studying the Collatz Conjecture in 2009. His other talents and interests include teaching, genealogy research, and board game design.

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